1 Introduction

The plasma around combination time is composed of thermal photon field and ideal (monotomic) "gases" electrons and proton "gases".

2 Photons

The Planck function specifies not just the energy distribution of the photons but also their absolute number per unit volume for given temperature. The photon field which is the CMB remains thermal — i.e. descibed by Planck function as the space expands. It is not immediately obvious to me that this has to be the case.

The Planck function in W m^{-3} Hz^{-1} is

$$I(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} \tag{1}$$

Dividing by the energy per photon $h\nu$ the photon density is

$$n(\nu) = \frac{8\pi \nu^2}{c^3} \frac{1}{e^{h\nu/kT} - 1} \tag{2}$$

The first term is the density of states and the second is the probablity to occupy each available state (the Bose-Einstein distribution since we are dealing with photons).

When we stretch space by factor 2 I think of it this way: i) each photon is stretched to $\nu/2$, ii) each band of photons now occupies only half the frequency interval so their number per Hz is doubled, iii) a given set of photons now occupy 8 times the space. The net effect of these three is the same as dividing T by 2.

The shape of the distribution is a trade off between ν^2 and the exponential. Dividing T by 2 shifts the peak of the distribution to $\nu/2$, and given the ν^2 to one quarter the peak height.

If it was anything other than ν^2 the photon field would not remain thermal.

3 Electrons and Protons

The electrons and protons are basically ideal gases. As space expands these gases expand adibatically - i.e. no heat enters or leaves the system.

For an adiabatic expansion

$$P_i V_i^{5/3} = P_f V_f^{5/3} \tag{3}$$

If we expand space by factor two $V_f = 8V_i$ so $P_f = P_i/32$.

Putting these into

$$\frac{P_i V_i}{T_i} = \frac{P_f V_f}{T_f} \tag{4}$$

Gives $T_f = T_i/4$. This seems to indicate that without interaction the temperature of the electrons and protons would diverge from that of the photons. The interaction is normally referred to as Thompson scattering which is the low energy limit of Compton scattering

$$\Delta\lambda = \frac{h}{m_e c}(1 - \cos\theta) \tag{5}$$

The shift is a fixed maxmimum $\Delta\lambda$ of 0.002 nm, so for optical photons the energy exchange with the electron is tiny.

4 Relative energy densities

The energy density of a thermal photon field is determined by temperature alone and is given by $\rho_r = \frac{4}{c}\sigma T^4$. For $T_{cmb} = 2.73$ K that gives 4.2×10^{-14} J m^{-3} (or 0.26 eV cm^{-3}). Assuming the Universe has expanded 1100 times since combination occured, and hence that the temperature was the same factor higher, the photon energy density then was 0.062 J m^{-3}

Assuming $h = 0.72$ and $\Omega_b = 0.05$ the absolute density today $\rho_b = 4.9 \times 10^{-28}$ kg m^{-3}. Given that mass energy density dilutes as the third power of the expansion at combination ρ_b was 6.5×10^{-19} kg m^{-3}. So assuming pure hydrogen the number density of electrons and protons was 3.9×10^8 m^{-3}. (Cf to the number density of particles in an ideal gas at standard temperature and pressure of 2.7×10^{25} m^{-3}.)

The mean kinetic energy per particle is $\frac{3}{2}kT$ so the kinetic energy density of the proton/electron gas was $\rho_{ke} = 4.8 \times 10^{-11}$ J m^{-3}. So the photons had more than 10^9 times as much "free energy" as the electrons and protons.