CMBFAST for spatially closed universes

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ABSTRACT

We extend the cosmological linear perturbation theory code CMBFAST to closed geometries. This completes the implementation of CMBFAST to all types of geometries and allows the user to perform an unlimited search in the parameter space of models. This will be specially useful for placing confidence limits on cosmological parameters from existing and future data. We discuss some of the technical issues regarding the implementation.

Subject headings: large-scale structure of universe; methods: numerical; methods: statistical

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1. Introduction

The study of the anisotropies in the Cosmic Microwave Background (CMB) is revolutionizing cosmology. Fluctuations across a range of angular scales have been detected and their precision in the measurement of the power spectra are improving steadily. Ground based and balloon born experiments are making effort to detect the acoustic features in the power spectrum predicted by inflationary models. In the near future the MAP satellite and a few years later the Planck satellite will be able to measure the CMB power spectrum with sufficient accuracy to constraint several cosmological parameters with a few percent accuracy percent accuracy (Jungman et al. 1996, Bond et al. 1997, Zaldarriaga et al. 1997).

There are two reasons that explain why the CMB is so powerful at constraining the cosmological parameters. First is the improvement in the experimental sensitivity that allows the detection of these small anisotropies with increasing precision. Second is our ability to accurately calculate the predictions for the different world models. These theoretical predictions can be made with high accuracy because the calculation of the power spectrum only involves linear perturbation theory. The same fact that makes the anisotropies difficult to detect makes them also easy to calculate.

The most widely used Boltzmann code to calculate anisotropy spectra, CMBFAST, was developed by the authors and has become a standard tool freely available to the community Seljak & Zaldarriaga 1996, Zaldarriaga, Seljak & Bertschinger 1998. The code is based on the integral solution of the transport equation for the photons, the Boltzmann equation. The algorithm is much more efficient than the traditional way of solving the hierarchy because it takes advantage of the large difference between the distance photons have traveled since recombination and the horizon at that time.

There have been several studies intended to constrain cosmological parameters with the existing CMB data Lineweaver 1998, Tegmark 1999, Dodelson & Knox 1999. These studies rely on the speed of CMBFAST to search a multidimensional parameter space for the best fitting model and the confidence regions for the parameters. They typically evaluate millions of models which would be prohibitely slow with a traditional Boltzmann code taking a hundred times longer than CMBFAST to run. So far the available versions of CMBFAST did not allow the user to explore models with spatially closed geometries and thus that part of parameter space has not been explored with this code (although closed universe Boltzmann codes do exist, see e.g. White & Scott 1996). Since the data are pushing us into the corner of parameter space with vanishing curvature, any meaningful error determination must also explore the models with positive curvature. It is therefore timely to make CMBFAST available also for this class of models.

A key part of the CMBFAST algorithm is being able to calculate efficiently the radial part of the eigenfunctions of the Laplacian, the ultraspherical Bessel functions. In our implementation of open models we calculate these functions by integrating their corresponding differential equation. The straightforward extension of this algorithm is not possible for closed models. In this paper we explain the technical difficulties that arise for closed models and present the solution that is implemented in the new version of CMBFAST (V3.0).

2. The line of sight method and a brief description of geometries

In spherical coordinates the unperturbed geometry is described by,

$$ds^2 = a^2(\tau)[-d\tau^2 + d\chi^2 + r^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{1}$$

The scale factor is $a(\tau)$, the coordinate τ is conformal time and the radial coordinate χ is related to the angular radius r by

$$r(\chi) \equiv \begin{cases} K^{-1/2} \sin K^{1/2}\chi, \ K > 0 \\ \chi, \ K = 0 \\ (-K)^{-1/2} \sinh(-K)^{1/2}\chi, \ K < 0. \end{cases} \tag{2}$$

with $K = H_0^2(\Omega_0 - 1)$, where Ω_0 includes the contribution from both matter and the cosmological constant. Note that with our choice of variables we have $d\tau = -d\chi$ along radial geodesics.

To calculate the CMB power spectrum C_l we need to evolve the perturbations in the different species (CDM, baryons, photons, neutrinos). The perturbations are so small that linear theory suffices. The standard method is first to expand all perturbations in eigenvalues of the Laplacian which evolve independently. In a flat spacetime this is a Fourier expansion. The wavenumber β labels the different eigenmodes. The fluctuations produced by each mode can be calculated separately and then all the contributions to the power are added,

$$C_l = (4\pi)^2 \int \beta d\beta \ P(\beta)|\Delta_{Tl}(\beta)|^2. \tag{3}$$

We have introduced $\Delta_{Tl}(\beta)$, the contribution by a mode of wavenumber β to the amplitude of multipole l of the temperature anisotropies. The primordial power spectrum is denoted by $P(\beta)$. Equation (3) says that to obtain the total power in multipole l we add the contributions from all modes labeled by β. We have written this sum as a continuous integral, which applies to flat and open models. For closed spatial hypersurfaces the eigenvalues of the Laplacian are discrete so the integral becomes a discrete sum over β with $K^{-1/2}\beta = 3, 4, 5, \cdots$. The first two modes are pure gauge modes and should be excluded from the sum Abbott & Schaefer 1986.

In CMBFAST the amplitudes Δ_{Tl} are obtained using the integral solution,

$$\Delta_{Tl} = \int_0^{\tau_0} d\tau \Phi_\beta^l(\chi) S(\beta, \tau). \tag{4}$$

The ultraspherical Bessel functions $\Phi_\beta^l(\chi)$ depend on the wavenumber β and the radial coordinate along the geodesic $\chi = \tau_0 - \tau$. The source $S(\beta, \tau)$ is a sum of the different terms that can generate temperature anisotropies. Among these are gravitational redshift, Doppler shift and intrinsic density fluctuation in the photon baryon fluid. The dominant contribution to the source is produced at the last scattering surface. The full expression for S together with the equivalent formulas for the CMB polarization can be found in Zaldarriaga, Seljak & Bertschinger 1998.

Equation (4) has several advantages over the more traditional way of calculating the anisotropies. The main one is that the sampling in β needed to compute accurately the source S (essentially the inverse of the sound horizon at recombination) is much smaller than that needed to obtain an accurate representation of Φ_β^l (approximately the inverse of the distance to the last scattering surface). In the traditional Boltzmann method one solves directly for Δ_{Tl} so one has to sample β with the inverse of the distance to the last scattering surface. Obtaining the source for each β involves solving the perturbation equations for the different species so a reduction in the number of equations to be solved greatly enhances the performance of the code.

In essence traditional Boltzmann codes spend most of their time computing Bessel functions. If we had a fast method for computing the Bessel functions we could use equation (4) to make a fast algorithm. This is what CMBFAST does. In flat geometries the ultraspherical Bessel functions reduce to ordinary spherical Bessel functions $j_l(\beta\chi)$ which are a function of a single variable $x = \beta\chi$. In the implementation of CMBFAST for flat models we use tables with the values of the spherical Bessel functions which are generated in advance and stored in disk.

For open model we are unable to tabulate the ultraspherical Bessel functions in advance because they no longer depend on only one variable. They depend separately on β and χ because the curvature radius of the universe introduces another scale that can be used to turn β and χ into two dimensionless numbers and the ultraspherical Bessel functions depend on these two numbers independently. In our implementation of CMBFAST for open models we choose to calculate the Bessel function by solving their differential equation,

$$\frac{d^2 u^l_\beta}{d\chi^2} + \left[\beta^2 - \frac{l(l+1)}{r(\chi)^2}\right] u^l_\beta = 0 \tag{5}$$

where $u^l_\beta(\chi) = r(\chi)\Phi^l_\beta(\chi)$.

Although at first this may not seem to be the most efficient method (for example, we could have used the recurrence relation for the ultraspherical Bessel functions instead), it is actually very efficient for our purposes. The point is that we need the function at a succession of neighboring points in time when evaluating equation (4). To get from one point to the next a Runge Kutta integrator only performs a handful of operations, as opposed to the recurrence relation calculation which has to start at sufficiently high l. It is true that recurrence relation gives the function value at all l, but since the final CMB spectra are so smooth we actually only use every 50th l or so, so this advantage of recurrence relation becomes less significant. Thus although the recurrence relation is the most efficient way to obtain a single value of the Bessel function, for our purposes solving the differential equation is a better method than the recurrence relation. Another method to calculate these Bessel functions based on the WKB approximation has been proposed (Kosowsky 1999). This method is again a fast way to obtain a single value of any Φ^l_β since it only involves a few operations per Φ^l_β call. However, for the integral solution which requires many calls at nearby arguments, the available implementation of these formulas does not appear to be faster than solving the differential equations.

3. Closed models

In Zaldarriaga, Seljak & Bertschinger 1998 we had developed the necessary formalism to solve the perturbation equations in both open and closed universes. We derived the Boltzmann hierarchy for both temperature and polarization and the integral solution. The other trivial differences between closed and open models like the discreteness of the eigenvalues of the Laplacian were also mentioned. The only problem not addressed was the calculation of the Bessel functions in closed models. We discuss this issue here.

Equation (5) is a second order differential equation and thus has two independent solutions. It can be viewed as the Schrödinger equation for a particle with energy $E = \beta^2$ in a potential $V = l(l+1)/r(\chi)^2$. Thus for χ such that $\beta r(\chi) > \sqrt{l(l+1)}$, where the "energy" is greater than the "potential", the solution is oscillatory. For $\beta r(\chi) < \sqrt{l(l+1)}$ there is a growing and a decaying solution, $u^l_\beta \propto r^{l+1}$ and $u^l_\beta \propto r^{-l}$, respectively. The growing solution corresponds to $\Phi^l_\beta(\chi)$. In order to obtain an accurate numerical convergence the integration needs to be started in the regime where $\Phi^l_\beta(\chi)$ is small, which in our case means starting the integration close to $\beta r(\chi) \approx l$. The equation then needs to be evolved in the direction of increasing χ until recombination, where $\chi \sim \tau_0$. The integration of $\Phi^l_\beta(\chi)$ therefore proceeds in the direction of decreasing time, opposite to the evolution of Boltzmann, fluid and Einstein equations. If one were to start the integration at early time and evolve the Bessel functions to small radial distances χ (i.e. present time) the integration would be numerically unstable, as the decaying mode $u^l_\beta \propto r^{-l}$ would increasingly contaminate the solution.

In closed models the angular distance $r(\chi)$ is periodic around $\theta \equiv K^{-1/2}\chi = \pi/2$. The potential $V(\chi)$ is a potential well. For parameters such that the maximum value of θ required in the evaluation of equation (4) is greater than $\pi/2$ the integration of the differential equation (5) becomes unstable, similarly to integrating in the direction of decreasing χ in the open model case. As the integration proceeds the solution becomes increasingly contaminated by the decaying solution which diverges at $\theta = \pi$. In figure 1 we show lines of constant θ in the $(\Omega_m, \Omega_\Lambda)$ plane for $z_{max} = 1000$. The region upwards of the $\theta = \pi/2$ line suffers from this instability.

We can use symmetry arguments analogous to those in quantum mechanics to solve the instability problem. The potential well $V(\chi)$ is symmetric around $\theta = \pi/2$ and thus the eigenfunctions are either symmetric or antisymmetric around this point. Just like in quantum mechanics we find that modes with even (odd) $\beta - l - 1$ are symmetric (antisymmetric). In our implementation of CMBFAST for closed models we use this fact to obtain the Bessel functions for regions with $\theta > \pi/2$. We integrate equation (5) up

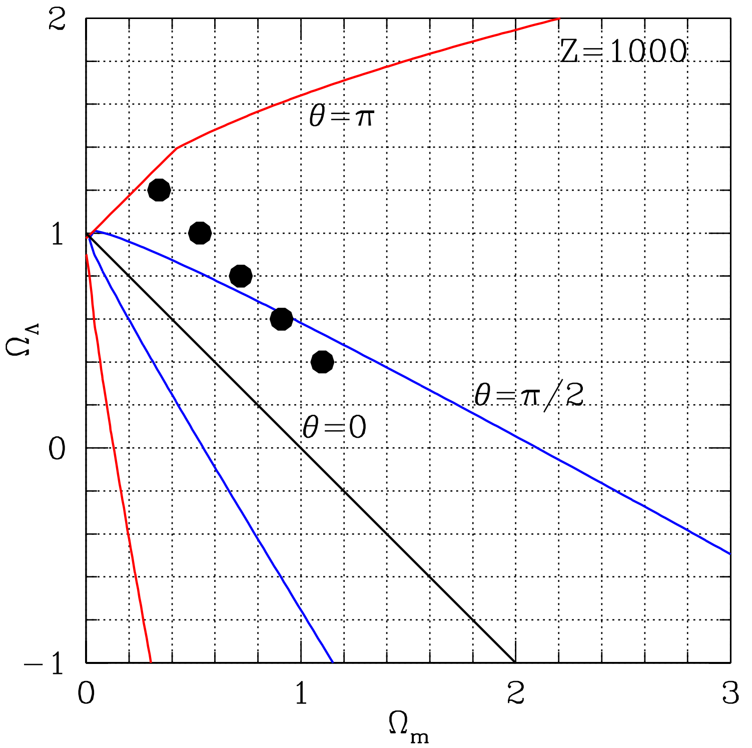

Fig. 1.— Contours of constant θ in the $(\Omega_m, \Omega_\Lambda)$ plane. The line $\theta = 0$ corresponds to flat models, with closed models above that line. In the region between $\theta = \pi/2$ and $\theta = \pi$ the calculation of the Bessel functions becomes unstable and symmetry can be used to determine their values in that regime. The dots show the positions of the models we show in figure 2. The two bottom lines also correspond to $\theta = \pi/2, \pi$ but in an open universe.

to $\theta = \pi/2$ and extend the solution further using the symmetry of the mode. In practice we apply the symmetry operation to the source term $S(\beta, \tau)$. We construct a symmetric $S^s = (S(\beta,\tau) + S(\beta,\tau'))/2$ and an antisymmetric $S^a = (S(\beta,\tau) - S(\beta,\tau'))/2$ version of the source. Here τ and τ' are times that have the same $r(\chi)$, related by $\tau' = K^{1/2}\pi - \tau$. We use one of those in (4) according to the parity of the mode being calculated and stop the time integration at the $\tau_{\pi/2} = \tau_0 - K^{1/2}\pi/2$ corresponding to $\theta = \pi/2$. We found this to be both a satisfactory solution of the instability and an efficient numerical algorithm.

In figure 2a we show the power spectra obtained using the new CMBFAST version for several closed models. We choose models with $\Omega_m h^2 = 0.237$, $\Omega_b h^2 = 0.013$ and varying $\Omega_\Lambda = 0.4, 0.6, 0.8, 1.0, 1.2$. The position of the models in the $(\Omega_m, \Omega_\Lambda)$ plane is shown in figure 1. The models we show all have the same values of $\Omega_m h^2$ and $\Omega_b h^2$, so the physics of the acoustic peaks is the same for all the models. They only differ in the angular diameter distance to the last scattering surface and their amplitude which was normalized to COBE. In figure 2b we rescale the l values so that a given physical scale at recombination corresponds to the same angular scale. This is achieved using $l \to l r'(\chi_R)/r(\chi'_R)$, where $r(\chi_R)$ is the radial comoving distance to recombination and primed variables correspond to the values in a fiducial model. We also renormalized the amplitude of the spectra so they are no longer COBE normalized. The agreement at high l overall is very good, which is a consistency check for the code.

As we approach $\theta = \pi$, $r(\chi) \to 0$, when we are able to see the antipode. As $\theta \to \pi$ the C_l spectra is

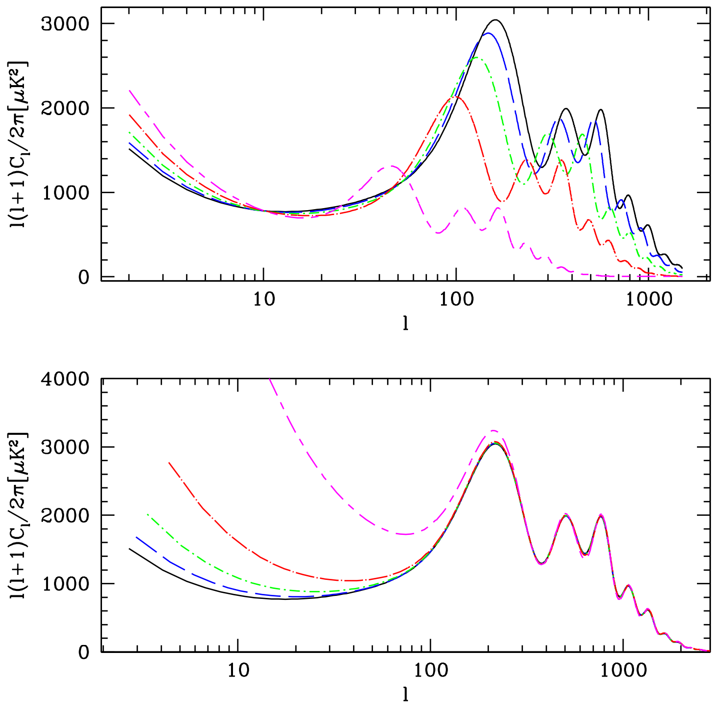

Fig. 2.— Sample of closed universe models calculated with the new version of CMBFAST. Panel b shows the same curves but rescaled in l and amplitude to match as closely as possible.

compressed so that the acoustic peaks appear at larger and larger scales. When $\theta = \pi$ all the CMB photons that arrive to us from every direction come form one single point, so there are no anisotropies. We have restricted the range of models implemented in CMBFAST to those with $\theta < \pi$.

In our previous implementations of CMBFAST we took advantage of the smoothness of the C_l spectra to calculate (4) for every 50th l in the region of the acoustic peaks. This was sufficient to obtain one percent accuracy in flat models and became even better for open model were the spectra are stretched further out in l. For closed models the situation is reversed because the spectra are compressed so this sampling can become insufficient. If the first acoustic peak occurs at $l \sim 130(100)$ as opposed to $l \sim 200$ for flat universes, the sampling every 50 ls introduces an error of 2% (5%).

4. Conclusions

We extended the CMBFAST code to the closed universes. The main technical issue in this implementation was the computation of ultraspherical Bessel functions, which was obtained by solving their differential equation jut like in the case of open models. We use the symmetry of the ultraspherical Bessel functions around $\theta = \pi/2$ to obtain their values at $\theta > \pi/2$ without integration. The code only works for models with $\theta < \pi$. The new version of CMBFAST for closed universes will allow the exploration of the full parameter space. The results of such an exploration will be presented elsewhere Tegmark & Zaldarriaga 1999.

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